Math for Game Programmers: Dual Numbers

Gino van den Bergen
gino@dtecta.com

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Introduction

- Dual numbers extend real numbers, similar to complex numbers.
- Complex numbers adjoin an element i, for which $i^2 = -1$.
- Dual numbers adjoin an element ε, for which $\varepsilon^2 = 0$.

Complex Numbers

- Complex numbers have the form

$$z = a + b\,i$$

where a and b are real numbers.

- a = real(z) is the real part, and
- b = imag(z) is the imaginary part.

Complex Numbers (cont'd)

- Complex operations pretty much follow rules for real operators:
- Addition:
$(a + b\,i) + (c + d\,i) = (a + c) + (b + d)\,i$
- Subtraction:
$(a + b\,i) - (c + d\,i) = (a - c) + (b - d)\,i$

Complex Numbers (cont'd)

- Multiplication:

$$(a + b\,i)\,(c + d\,i) = (ac - bd) + (ad + bc)\,i$$

- Products of imaginary parts feed back into real parts.

Dual Numbers

- Dual numbers have the form

$$z = a + b\,\varepsilon$$

similar to complex numbers.

- a = real(z) is the real part, and
- b = dual(z) is the dual part.

Dual Numbers (cont'd)

- Operations are similar to complex numbers, however since $\varepsilon^2 = 0$, we have:

$$(a + b\,\varepsilon)\,(c + d\,\varepsilon) = (ac + 0) + (ad + bc)\varepsilon$$

- Dual parts do not feed back into real parts!

Dual Numbers (cont’d)

- The real part of a dual calculation is independent of the dual parts of the inputs.

- The dual part of a multiplication is a “cross” product of real and dual parts.

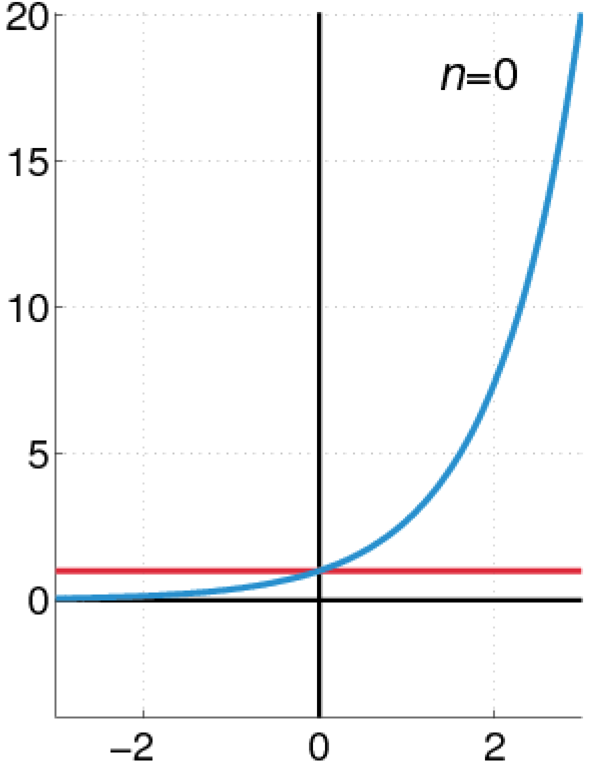
Taylor Series

- Any value $f(a + h)$ of a smooth function f can be expressed as an infinite sum:

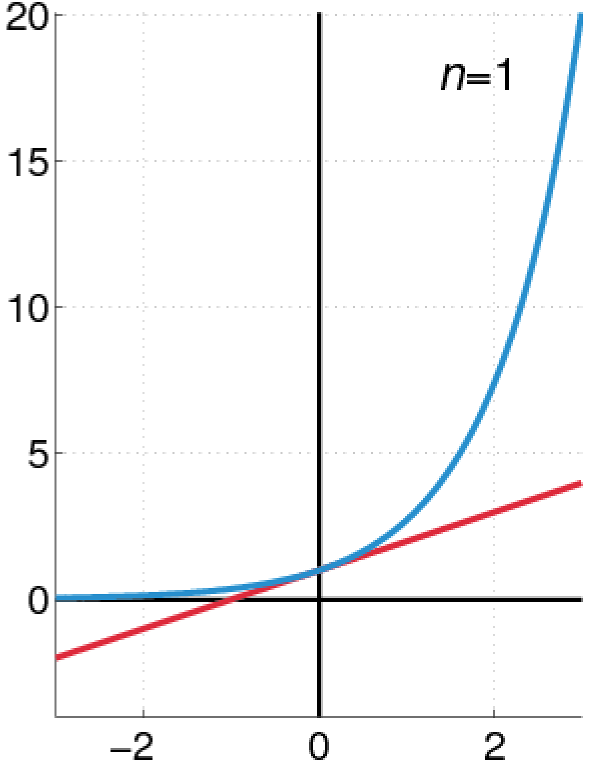
$$f(a+h) = f(a) + \frac{f'(a)}{1!}h + \frac{f''(a)}{2!}h^2 + \cdots$$

where f', f'', ..., $f^{(n)}$ are the first, second, ..., n-th derivative of f.

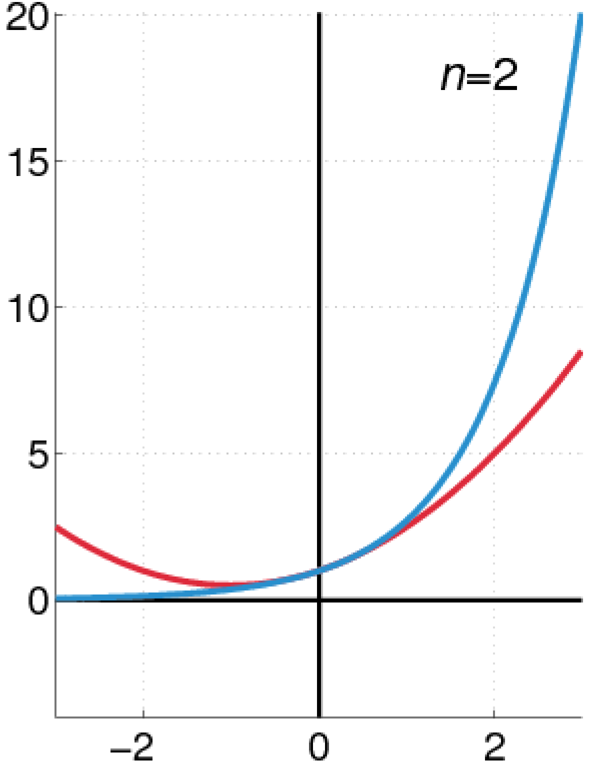
Taylor Series Example



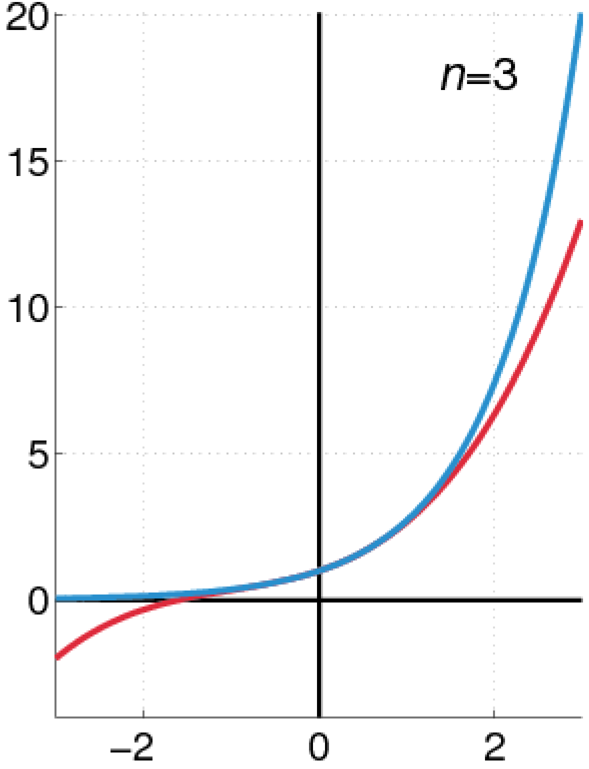
Taylor Series Example



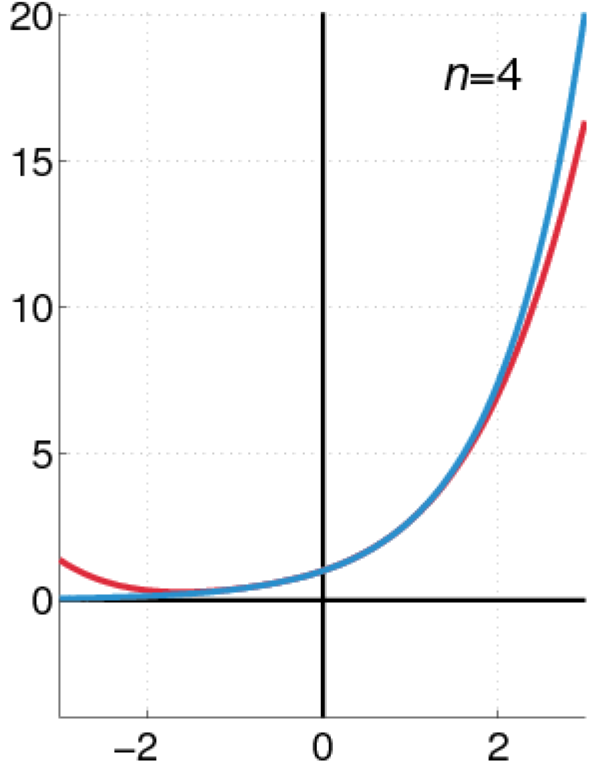
Taylor Series Example



Taylor Series Example



Taylor Series Example



Taylor Series and Dual Numbers

- For *f*(*a* + *b* ε), the Taylor series is:

$$f(a+b\varepsilon) = f(a) + \frac{f'(a)}{1!} b\varepsilon + \ldots 0$$

- All second- and higher-order terms vanish!
- We have a closed-form expression that holds the function and its derivative.

Real Functions on Dual Numbers

- Any differentiable real function f can be extended to dual numbers, as:

$$f(a + b\,\varepsilon) = f(a) + b\,f'(a)\,\varepsilon$$

- For example,

$$sin(a + b\,\varepsilon) = sin(a) + b\cos(a)\,\varepsilon$$

Automatic Differentiation

- Add a unit dual part to the input value of a real function.
- Evaluate function using dual arithmetic.
- The output has the function value as real part and the derivate's value as dual part:

$$f(a + \varepsilon) = f(a) + f'(a)\,\varepsilon$$

How does it work?

- Check out the product rule of differentiation: $(f \cdot g)' = f \cdot g' + f' \cdot g$
- Notice the “cross” product of functions and their derivatives.
- Recall that $(a + a'\varepsilon)(b + b'\varepsilon) = ab + (ab' + a'b)\varepsilon$

Automatic Differentiation in C++

- We need some easy way of extending functions on floating-point types to dual numbers…
- …and we need a type that holds dual numbers and offers operators for performing dual arithmetic.

Extension by Abstraction

- C++ allows you to abstract from the numerical type through:
  - Typedefs
  - Function templates
  - Constructors and conversion operators
  - Overloading
  - Traits class templates

Abstract Scalar Type

- Never use built-in floating-point types, such as `float` or `double`, explicitly.
- Instead use a type name, e.g. `Scalar`, either as template parameter or as typedef,

```
typedef float Scalar;
```

Constructors

- Built-in types have constructors as well:
  - Default: `float() == 0.0f`
  - Conversion: `float(2) == 2.0f`
- Use constructors for defining constants, e.g. use `Scalar(2)`, rather than `2.0f` or `(Scalar)2`.

Overloading

- Operators and functions on built-in types can be overloaded in numerical classes, such as `std::complex`.
- Built-in types support operators: `+,-,*,/`
- ...and functions: `sqrt, pow, sin, …`
- NB: Use `<cmath>` rather than `<math.h>`. That is, use `sqrt` NOT `sqrtf` on floats.

Traits Class Templates

- Type-dependent constants, such as the machine epsilon, are obtained through a traits class defined in `<limits>`.
- Use `std::numeric_limits<Scalar>::epsilon()` rather than `FLT_EPSILON` in C++.
- Either specialize `std::numeric_limits` for your numerical classes or write your own traits class.

Example Code (before)

```
float smoothstep(float x)
{
    if (x < 0.0f)
        x = 0.0f;
    else if (x > 1.0f)
        x = 1.0f;
    return (3.0f - 2.0f * x) * x * x;
}
```

Example Code (after)

```
template <typename T>
T smoothstep(T x)
{
    if (x < T())
        x = T();
    else if (x > T(1))
        x = T(1);
    return (T(3) - T(2) * x) * x * x;
}
```

Dual Numbers in C++

- C++ has a standard class template `std::complex<T>` for complex numbers.
- We create a similar class template `Dual<T>` for dual numbers.
- `Dual<T>` defines constructors, accessors, operators, and standard math functions.

Dual<T>

```
template <typename T>
class Dual
{
…
private:
        T mReal;
        T mDual;
};
```

Dual<T>: Constructor

```
template <typename T>
Dual<T>::Dual(T real = T(), T dual = T())
    : mReal(real)
    , mDual(dual)
{}

…
Dual<Scalar> z1; // zero initialized
Dual<Scalar> z2(2); // zero dual part
Dual<Scalar> z3(2, 1);
```

Dual<T>: operators

```
template <typename T>
Dual<T> operator*(Dual<T> a, Dual<T> b)
{
  return Dual<T>(
            a.real() * b.real(),
            a.real() * b.dual() +
               a.dual() * b.real()
         );
}
```

Dual<T>: Standard Math

```
template <typename T>
Dual<T> sqrt(Dual<T> z)
{
    T tmp = sqrt(z.real());
    return Dual<T>(
               tmp,
               z.dual() * T(0.5) / tmp
           );
}
```

Curve Tangent

- For a 3D curve

$$\mathbf{p}(t) = (x(t), y(t), z(t)), \quad \text{where} \quad t \in [a, b]$$

The tangent is

$$\frac{\mathbf{p}'(t)}{\|\mathbf{p}'(t)\|}, \quad \text{where} \quad \mathbf{p}'(t) = (x'(t), y'(t), z'(t))$$

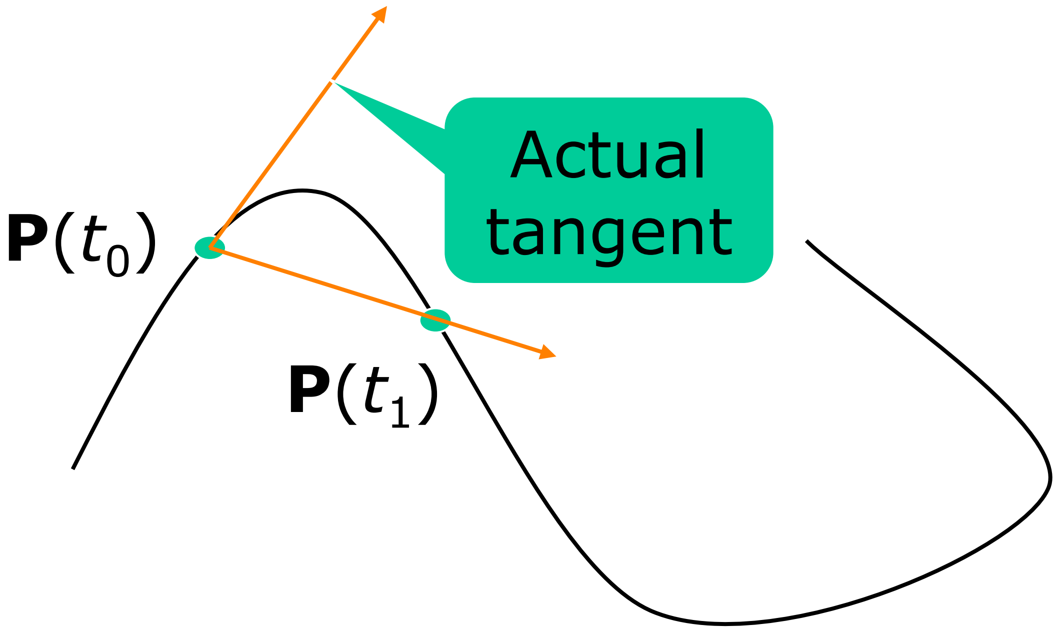
Curve Tangent

- Curve tangents are often computed by approximation:

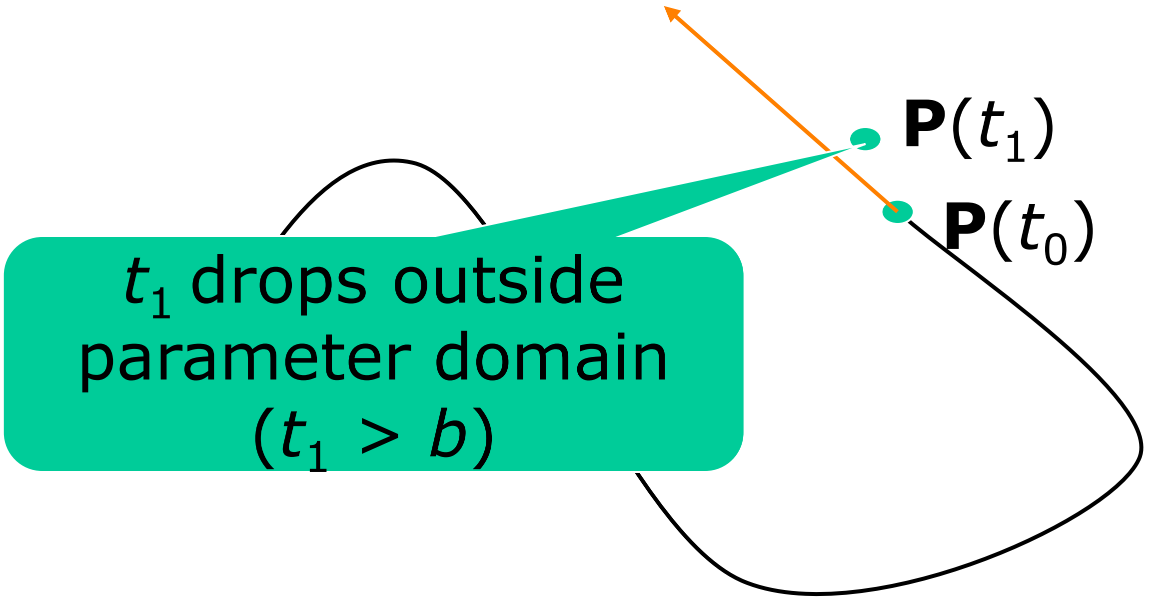
$$\frac{\mathbf{p}(t_1)-\mathbf{p}(t_0)}{\left\|\mathbf{p}(t_1)-\mathbf{p}(t_0)\right\|}, \quad \text{where} \quad t_1 = t_0 + h$$

for tiny values of *h*.

Curve Tangent: Bad #1



Curve Tangent: Bad #2



Curve Tangent: Duals

- Make a curve function template using a class template for 3D vectors:

```
template <typename T>
Vector3<T> curveFunc(T x);
```

Curve Tangent: Duals (cont'd)

- Call the curve function using a dual number `x = Dual<Scalar>(t, 1)`, (add *ε to parameter* `t`):

```
Vector3<Dual<Scalar> > y =
      curveFunc(Dual<Scalar>(t, 1));
```

Curve Tangent: Duals (cont'd)

- The real part is the evaluated position:

```
Vector3<Scalar> position = real(y);
```

- The normalized dual part is the tangent at this position:

```
Vector3<Scalar> tangent =
    normalize(dual(y));
```

Line Geometry

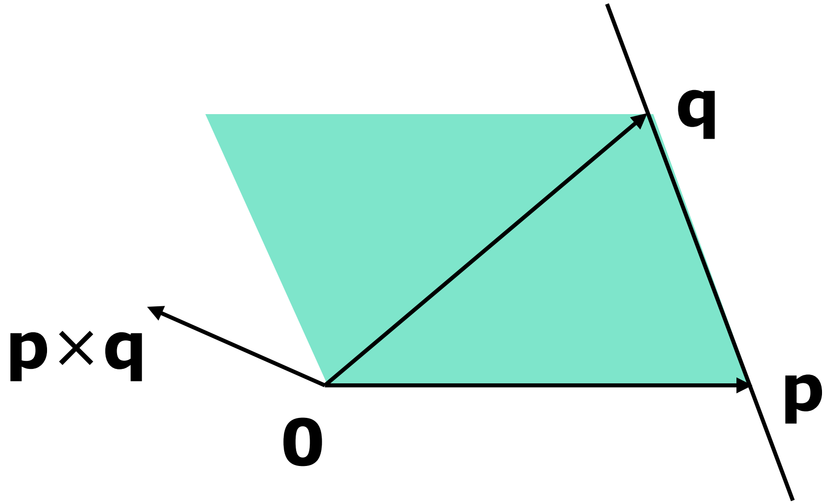
- The line through points **p** and **q** can be expressed explicitly as:

$$\mathbf{x}(t) = \mathbf{p} + (\mathbf{q} - \mathbf{p})t, \text{ and}$$

- Implicitly, as a set of points **x** for which:

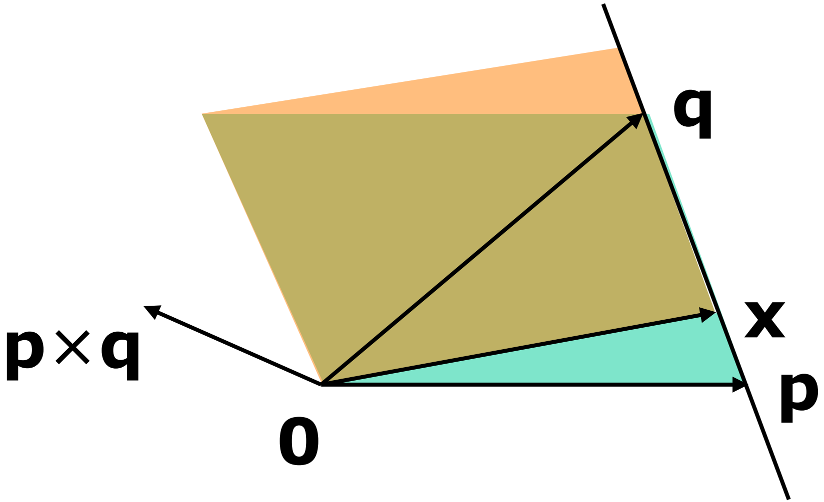
$$(\mathbf{q} - \mathbf{p}) \times \mathbf{x} + \mathbf{p} \times \mathbf{q} = \mathbf{0}$$

Line Geometry



p × **q** is orthogonal to the plane **opq**, and its length is equal to the area of the parallellogram spanned by **p** and **q**

Line Geometry



All points **x** on the line **pq** span with **q** – **p** a parallellogram that has the same area and orientation as the one spanned by **p** and **q**.

Plücker Coordinates

- Plücker coordinates are 6-tuples of the form $(u_x, u_y, u_z, v_x, v_y, v_z)$, where

$$\mathbf{u} = (u_x, u_y, u_z) = \mathbf{q} - \mathbf{p}, \quad \textit{and}$$

$$\mathbf{v} = (v_x, v_y, v_z) = \mathbf{p} \times \mathbf{q}$$

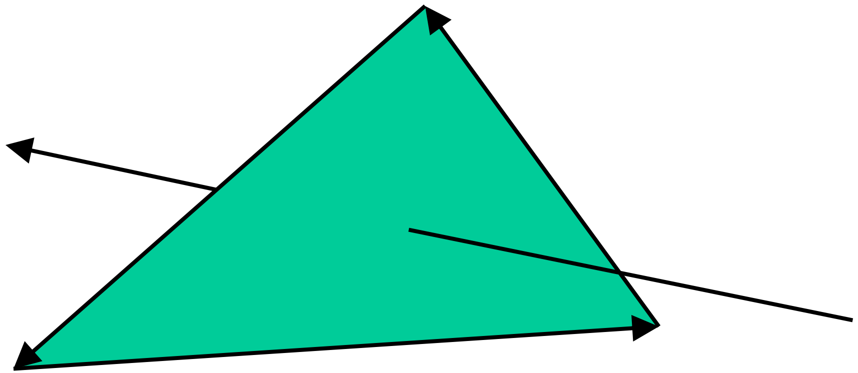
Plücker Coordinates (cont'd)

- For $(\mathbf{u}_1 : \mathbf{v}_1)$ and $(\mathbf{u}_2 : \mathbf{v}_2)$ directed lines, if

$$\mathbf{u}_1 \bullet \mathbf{v}_2 + \mathbf{v}_1 \bullet \mathbf{u}_2 \quad \text{is}$$

zero: the lines intersect
positive: the lines cross right-handed
negative: the lines cross left-handed

Triangle vs. Ray



If the signs of permuted dot products of the ray and edges are all equal, then the ray intersects the triangle.

Plücker Coordinates and Duals

- Dual 3D vectors conveniently represent Plücker coordinates:

```
Vector3<Dual<Scalar> >
```

- For a line (**u**:**v**), **u** is the real part and **v** is the dual part.

Dot Product of Dual Vectors

- The dot product of dual vectors $\mathbf{u}_1 + \mathbf{v}_1\varepsilon$ and $\mathbf{u}_2 + \mathbf{v}_2\varepsilon$ is a dual number z, for which

$$\mathrm{real}(z) = \mathbf{u}_1 \bullet \mathbf{u}_2, \text{ and}$$
$$\mathrm{dual}(z) = \mathbf{u}_1 \bullet \mathbf{v}_2 + \mathbf{v}_1 \bullet \mathbf{u}_2$$

- The dual part is the permuted dot product

Angle of Dual Vectors

- For **a** and **b** dual vectors, we have

$$\theta + d\varepsilon = \arccos\left(\frac{\mathbf{a} \bullet \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}\right)$$

where θ is the angle and d is the signed distance between the lines **a** and **b**.

Translation

- Translation of lines only affects the dual part. Translation of line **pq** over **c** gives:
- Real: $(\mathbf{q} + \mathbf{c}) - (\mathbf{p} + \mathbf{c}) = \mathbf{q} - \mathbf{p}$
- Dual: $(\mathbf{p} + \mathbf{c}) \times (\mathbf{q} + \mathbf{c})$
  $= \mathbf{p} \times \mathbf{q} + \mathbf{c} \times (\mathbf{q} - \mathbf{p})$
- $\mathbf{q} - \mathbf{p}$ pops up in the dual part!

Rotation

- Real and dual parts are rotated in the same way. For a rotation matrix $\mathbf{R}$:
- Real: $\mathbf{Rq} - \mathbf{Rp} = \mathbf{R}(\mathbf{q} - \mathbf{p})$
- Dual: $\mathbf{Rp} \times \mathbf{Rq} = \mathbf{R}(\mathbf{p} \times \mathbf{q})$
- The latter holds for rotations only! That is, $\mathbf{R}$ performs no scaling or reflection.

Rigid-Body Transform

- For rotation matrix **R** and translation vector **c**, the dual 3×3 matrix **M** with

real(**M**) = **R**, and

$$\text{dual}(\mathbf{M}) = [\mathbf{c}]_\times \mathbf{R} = \begin{bmatrix} 0 & -c_z & c_y \\ c_z & 0 & -c_x \\ -c_y & c_x & 0 \end{bmatrix} \mathbf{R}$$

maps Plücker coordinates to the new reference frame.

Screw Theory

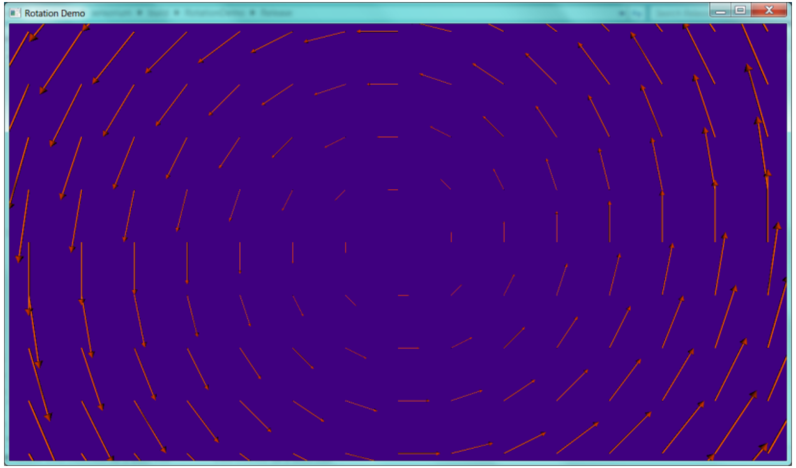
- A screw motion is a rotation about a line and a translation along the same line.

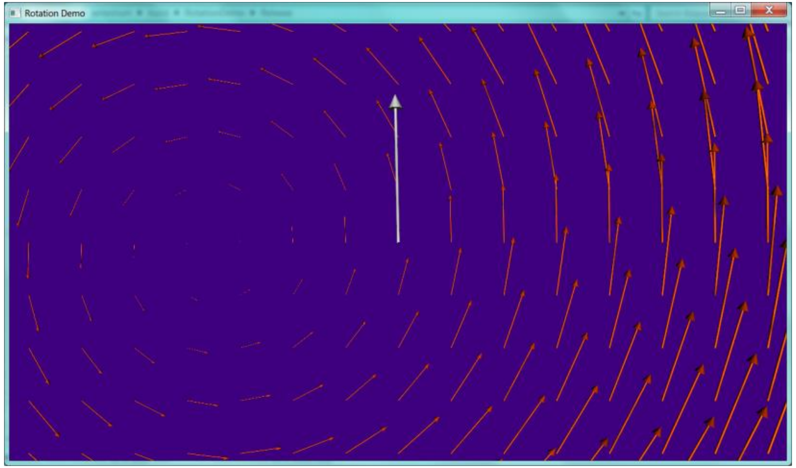
- "*Any rigid body displacement can be defined by a screw motion*." (Chasles)

Chasles' Theorem (Sketchy Proof)

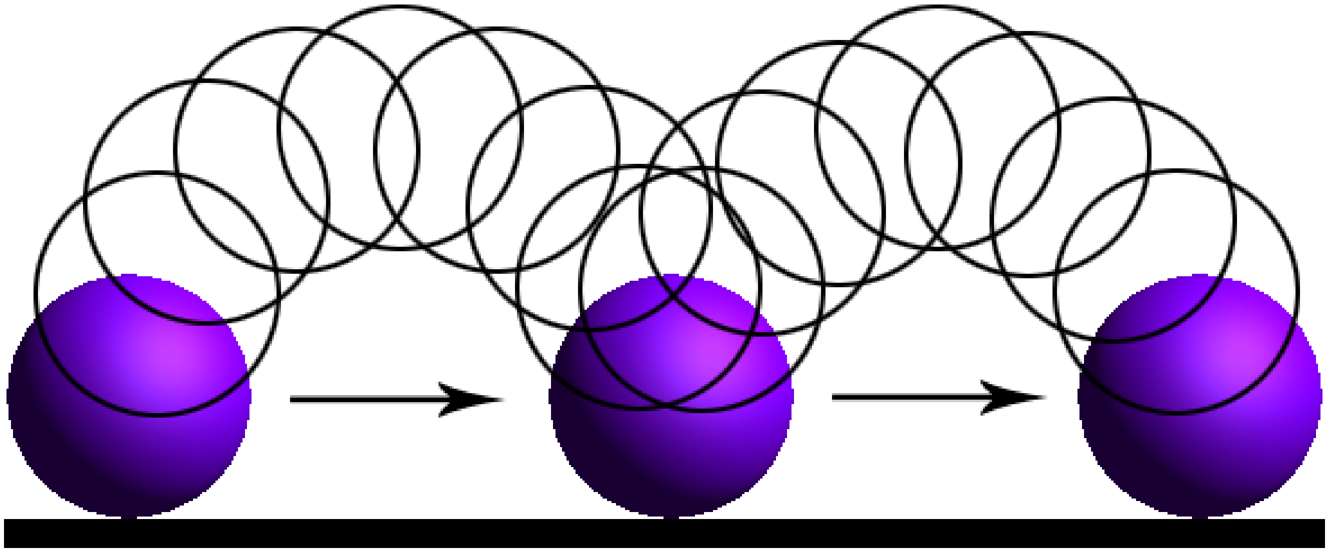
- Decompose translation into a term along the line and a term orthogonal to the line.
- Translation orthogonal to the axis of rotation offsets the axis.
- Translation along the axis does not care about the position of the axis.

Translations Orthogonal to Axis





Example: Rolling Ball



Dual Quaternions

- Unit dual quaternions represent screw motions.
- The rigid body transform over a unit quaternion **q** and vector **t** is:

$$\mathbf{q} + \frac{1}{2}\mathbf{t}\mathbf{q}\varepsilon$$

Here, **t** is a quaternion with zero scalar part.

Where is the Screw?

- A unit dual quaternion can be written as

$$\cos\left(\frac{\theta + d\varepsilon}{2}\right) + \sin\left(\frac{\theta + d\varepsilon}{2}\right)(\mathbf{u} + \mathbf{v}\varepsilon)$$

where θ is the rotation angle, d, the translation distance, and $\mathbf{u} + \mathbf{v}\varepsilon$, the line given in Plücker coordinates.

Rigid-Body Transform Revisited

- Similar to 3D vectors, Plücker coordinates can be transformed using dual quaternions.
- The mapping of a dual vector **v** according to a screw motion **q** is

$$\mathbf{v}' = \mathbf{q}\,\mathbf{v}\,\mathbf{q}^*$$

Traditional Skinning

- Bones are defined by transformation matrices $\mathbf{T}_i$ relative to the rest pose.
- Each vertex is transformed as

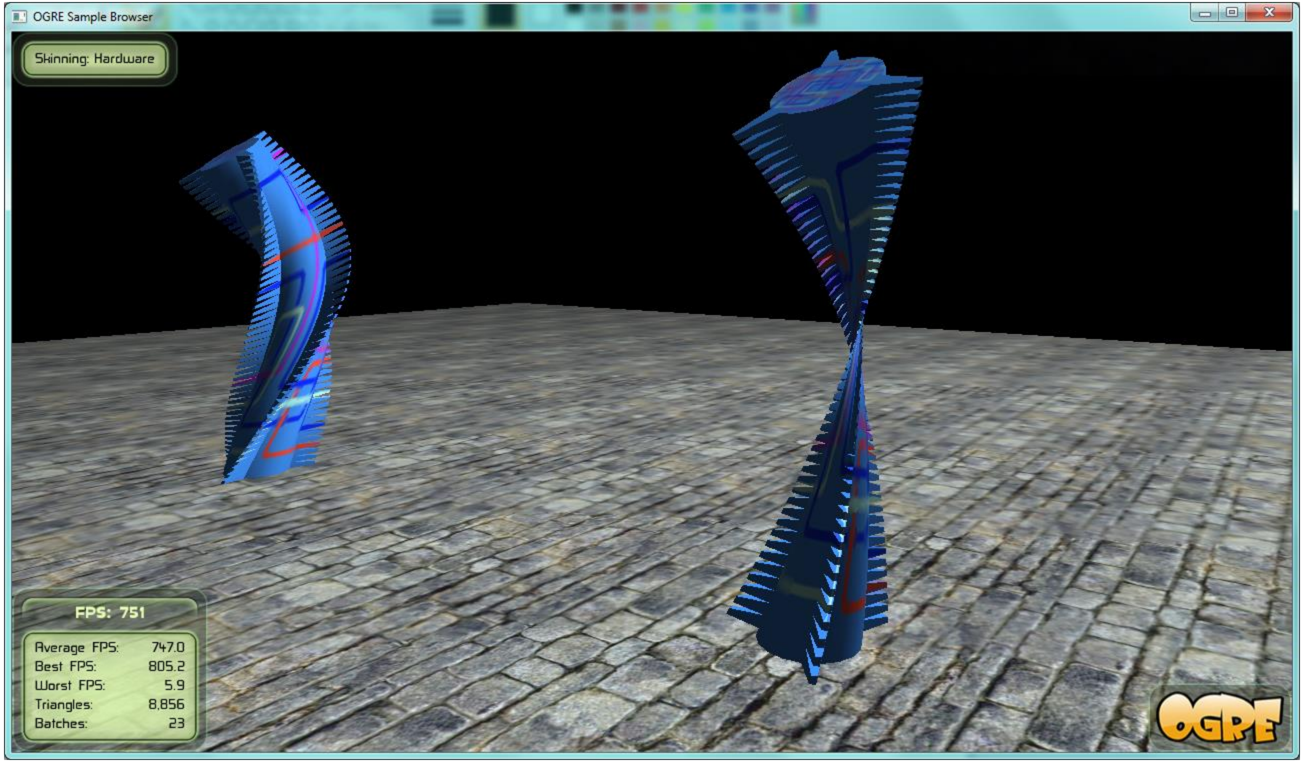
$$\mathbf{p}' = \lambda_1 \mathbf{T}_1 \mathbf{p} + \cdots + \lambda_n \mathbf{T}_n \mathbf{p} = (\lambda_1 \mathbf{T}_1 + \cdots + \lambda_n \mathbf{T}_n)\mathbf{p}$$

Here, λ_i are blend weights.

Traditional Skinning (cont'd)

- A weighted sum of matrices is not necessarily a rigid-body transformation.
- Most notable artifact is "candy wrapper": The skin collapses while transiting from one bone to the other.

Candy Wrapper



Dual Quaternion Skinning

- Use a blend operation that always returns a rigid-body transformation.
- Several options exists. The simplest one is a normalized lerp of dual quaternions:

$$\mathbf{q} = \frac{\lambda_1 \mathbf{q}_1 + \cdots + \lambda_n \mathbf{q}_n}{\left\| \lambda_1 \mathbf{q}_1 + \cdots + \lambda_n \mathbf{q}_n \right\|}$$

Dual Quaternion Skinning (cont'd)

- Can the weighted sum of dual quaternions ever get zero?
- Not if all dual quaternions lie in the same hemisphere.
- Observe that $\mathbf{q}$ and $-\mathbf{q}$ are the same pose. If necessary, negate each $\mathbf{q}_i$ to dot positively with $\mathbf{q}_0$.

Further Uses

- **Motor Algebra**: Linear and angular velocity of a rigid body combined in a dual 3D vector.
- **Spatial Vector Algebra**: Featherstone uses 6D vectors for representing velocities and forces in robot dynamics.

Conclusions

- Abstract from numerical types in your C++ code.
- Differentiation is easy, fast, and exact with dual numbers.
- Dual numbers have other uses as well. Explore yourself!

References

- D. Vandevoorde and N. M. Josuttis. *C++ Templates: The Complete Guide.* Addison-Wesley, 2003.
- K. Shoemake. *Plücker Coordinate Tutorial.* *Ray Tracing News, Vol. 11, No. 1*
- R. Featherstone. *Robot Dynamics Algorithms.* Kluwer Academic Publishers, 1987.
- L. Kavan et al. Skinning with dual quaternions. *Proc. ACM SIGGRAPH Symposium on Interactive 3D Graphics and Games*, 2007

Thank You!

- For sample code, check out free* MoTo C++ template library on:

`https://code.google.com/p/motion-toolkit/`

(*) gratis (as in “free beer”) and libre (as in “free speech”)